# MATH 530 Qualifying Exam

August 2009 (S. Bell, A. Eremenko)

*Each problem is worth 25 points*

**1.** Compute $\int_0^\infty \sin(x^2)\,dx$ by integrating $e^{-z^2}$ around the contour that follows the real axis from the origin to $R$, then follows the circle $Re^{it}$ as $t$ ranges from $t = 0$ to $t = \pi/4$, and then follows a line back from $Re^{i\pi/4}$ to the origin. Let $R$ tend to infinity.

You may use the fact that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ without proof.

**2.** Suppose $a_1, a_2, \ldots, a_N$ are distinct non-zero complex numbers and let $\Omega$ denote the domain obtained by removing the union of the closed line segments joining each of the points $a_k$ to the origin, $k = 1, 2, \ldots, N$. Prove that there is an analytic function $f(z)$ on $\Omega$ such that

$$f(z)^N = \prod_{k=1}^{N}(z - a_k).$$

**3.** For a closed contour $\gamma$, let

$$F(z) = \int_\gamma \frac{1}{w - z}\,dw$$

for $z$ in the open set $\Omega$ equal to the complex plane minus the trace of $\gamma$. Flesh out the details in the following standard argument.

a) Use the definition of the complex derivative and careful estimates to prove that

$$F'(z) = \int_\gamma \frac{1}{(w - z)^2}\,dw.$$

b) Use the formula for $F'$ in part (a) to explain why $F'(z) \equiv 0$ on $\Omega$.

c) Prove that $F' \equiv 0$ implies that $F$ is locally constant on $\Omega$.

d) Using only the fact that $F$ is locally constant and the definition of a connected open set, prove that $F(z)$ must be constant on each connected component of $\Omega$.

e) Give an example of a piecewise $\mathcal{C}^1$ curve $\gamma$ such that the resulting open set $\Omega$ has infinitely many connected components.

**4.** Suppose that $f(z)$ is merely a *continuous* complex valued function near the point $z_0$. Let $C_r(z_0)$ denote the circle of radius $r$ centered at $z_0$. Prove that

$$\lim_{r \to 0+} \int_{C_r(z_0)} \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0).$$

**5.** Suppose that $f$ is analytic on a domain $\Omega$ and satisfies $|f(z) + 2| < 2$ on $\Omega$. If $a$ and $b$ are points in $\Omega$, and if $f(a) = -1 - i$ and $f(b) = -1 + i$, compute

$$\int_\gamma \frac{f'(z)}{f(z)} \, dz$$

for any curve $\gamma$ in $\Omega$ that starts at $a$ and ends at $b$.

**6.** Prove that a positive harmonic function on the whole complex plane must be constant.

**7.** Suppose that $f$ is analytic on the unit disc minus the origin and that $\operatorname{Re} f \geq 0$. What kind of isolated singularity can $z = 0$ be? What can be said if we only know that $f(z) \notin [-1, 1]$?

**8.** Suppose that $f$ is a one-to-one conformal mapping of the unit disc onto the interior of a triangle. Is $f$ a linear fractional transformation? Prove your assertion.