MATH 530 Qualifying Exam

August 2008 (S. Bell, A. Eremenko)

Each problem is worth 25 points

1. Suppose u and v are real valued harmonic functions on a domain Ω.

a) *(10 pts.)* If u and v agree on a set with a limit point in Ω, does it follow that $u = v$ on all of Ω? Explain.
b) *(15 pts.)* If u and v satisfy the Cauchy-Riemann equations on a set with a limit point in Ω, does it follow that $u + iv$ is analytic on Ω? Explain.

2. Suppose that f is continuous on $\{z : |z| \leq 1\}$ and analytic in $\{z : |z| < 1\}$.

a) *(15 pts.)* Prove that if f vanishes on $\{z : z = e^{i\theta}, 0 \leq \theta \leq \frac{\pi}{2}\}$, then $f \equiv 0$ on the unit disc. Hint: Where do $f(i^n z)$ vanish for $n = 1, 2, 3$?
b) *(10 pts.)* Does the same result hold for harmonic functions? Explain.

3. Suppose f and g are analytic in a neighborhood of a point a, and suppose g has a simple zero at a. Find a formula for the residue of $\dfrac{f(z)}{g(z)^2}$ at a in terms of $f(a)$ and derivatives of f and g at a.

4. Let $\log z$ denote the branch of a complex logarithm with branch cut along the negative imaginary axis such that $-\frac{\pi}{2} < \operatorname{Im} \log z < \frac{3\pi}{2}$. Let C_r denote the curve parameterized by $z(t) = re^{it}$, $0 \leq t \leq \pi$. Prove that $\displaystyle\int_{C_r} \frac{\log z}{(z^2+1)^2}\, dz$ tends to zero as r tends to ∞ *and* as r tends to zero. Compute

$$\int_0^\infty \frac{\log x}{(x^2+1)^2}\, dx.$$

5. Suppose that f is analytic in a neighborhood of the closed unit disc $\{z : |z| \leq 1\}$. If $|f(z)| < 1$ when $|z| = 1$, prove that there must be at least one point z_0 with $|z_0| < 1$ such that $f(z_0) = z_0$.

6. Find a one-to-one conformal mapping from the region $\{z : \operatorname{Re} z > 0\} - [0, 1]$ onto the unit disc.

7. Notation: $D_r(a)$ denotes the open disc of radius r about a and $C_r(a)$ denotes the counterclockwise circle about a parametrized by $z(t) = a + re^{it}$, $0 \leq t \leq 2\pi$.

Let Ω be the domain given by $D_5(0)$ minus the two closed discs $\overline{D_1(-2)}$ and $\overline{D_1(2)}$. Suppose f is analytic on Ω and

$$\int_{C_2(-2)} f\ dz = \pi \qquad \text{and} \qquad \int_{C_2(2)} f\ dz = e.$$

Draw Ω and draw a closed contour γ in Ω such that

$$\mathrm{Ind}_\gamma(-2) = -2 \qquad \text{and} \qquad \mathrm{Ind}_\gamma(2) = 1.$$

What is the value of $\int_\gamma f\,dz$? Explain.

8. There is a famous conformal mapping $f(z)$ of the unit square

$$\mathcal{S} = \{z : 0 < \mathrm{Re}\, z < 1, 0 < \mathrm{Im}\, z < 1\}$$

onto the upper half plane which extends continuously up to the boundary of $\mathcal{S}$ and which maps the boundary of S into the real line. Assume that $f(0) = 0$. Prove that f extends analytically to a neighborhood of the origin and that the extension has a double zero at zero.