QUALIFYING EXAMINATION

JANUARY 2000

MATH 530 - Prof. Bell

1. *(20 pts)* Prove that $\ln|z|$ cannot have a harmonic conjugate on the domain $\{z : 1 < |z| < 2\}$.

2. *(20 pts)* Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers in the unit disk. What can you say about the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ if $|a_n| \to 1$ as $n \to \infty$? What can you say about the radius of convergence if the set $\{a_n\}$ is dense in the unit disk?

3. *(20 pts)* Suppose that γ_1 and γ_2 are two continuously differentiable curves that cross at a point z_0 in the complex plane and that their tangent vectors make an angle α at z_0. If the two curves are contained in the zero set of a harmonic function that is not identically zero, what are the possible values of α? If $\alpha = 0$, what can you say about the two curves near z_0?

4. *(20 pts)* Calculate

$$\int_0^{\infty} \frac{\ln x}{(x^3+1)}\, dx.$$

by integrating a meromorphic function around a contour γ described as follows. Let $\alpha = 2\pi/3$. The contour γ follows the real axis from the ϵ to R, then follows the circle Re^{it} from $t = 0$ to $t = \alpha$, and then follows a line back to $\epsilon e^{i\alpha}$, then follows the circle ϵe^{it} back to ϵ.

5. *(20 pts)* Let z_n be a sequence of distinct non-zero complex numbers such that $z_n \to \infty$ as $n \to \infty$, and let m_n be a sequence of positive integers. Let g be a meromorphic function on the plane having simple poles with residue m_n at z_n and having no other poles. If $z \neq z_n$ for all n, let γ_z be any path from 0 to z which avoids the set $\{z_n\}$. Define

$$f(z) = \exp\left(\int_{\gamma_z} g(\zeta)\, d\zeta\right).$$

Prove that $f(z)$ is independent of the choice of γ_z (although the integral itself might not be). Prove that f is analytic on the complement of $\{z_n\}$, that f has removable singularities at each point z_n, and that the extension of f has a zero of order m_n at z_n.

You have shown that the Weierstraß Theorem follows from the Mittag-Leffler Theorem.