MATH 530 Qualifying Exam

January 1998

Notation: $D_r(a)$ denotes the disk, $\{z \in \mathbb{C} : |z-a| < r\}$.

1. *(15 pts)* Evaluate the integral $\displaystyle\int_{-\infty}^{0} \frac{x^2}{x^4+x^2+1}\,dx.$

2. *(15 pts)* Find a one-to-one analytic map from $D_1(0) \cap \{x+iy : x, y > 0\}$ onto $D_1(0)$.

3. *(25 pts)* Let $\mathcal{F}$ denote the set of analytic functions f on $D_1(0)$ such that $|f(z)| < 1$ for all $z \in D_1(0)$, $f(0) = 0$, and $f'(0) = 0$. Prove that if $f \in \mathcal{F}$, then $|f(z)| \le |z|^2$ for all $z \in D_1(0)$. Let $M = \sup\{|f''(0)| : f \in \mathcal{F}\}$. Find all functions, if any, in $\mathcal{F}$ such that $|f''(0)| = M$.

4. *(15 pts)* How many zeroes does the polynomial

$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.

5. *(15 pts)* Prove that a harmonic function cannot have an isolated zero.

6. *(15 pts)* Let $C_1(0)$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and let f be a function that is analytic on $D_r(0)$ for some $r > 1$. Prove that if $f(C_1(0)) \subset C_1(0) \setminus \{1\}$, then f is a constant function.