MA 530
Qualifying Examination
January 1997

Name:____

1. Let r be the radius of convergence of $\sum a_n z^n$, and ρ be the radius of convergence of $\sum b_n z^n$. Set $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$ for $n = 0, 1, \ldots$. What can the radius of convergence R of the series $\sum c_n z^n$ be? Describe all possibilities.

2. For what R does there exist a one-to-one analytic map f_R from the left half-plane $\{z \in \mathbb{C} : \Re e\ z < 0\}$ onto the disk $\mathcal{D}_R = \{z \in \mathbb{C} : |z| < R\}$ such that $f_R(-3) = 0$ and $f_R'(-3) = 1 + i$? Find f_R for those R for which it exists and prove that for all other R's it does not exist.

3. How many zeros does the polynomial $z^{10} - z^3 + 3$ have in the first quadrant?

4. Find and classify all isolated singularities in $\overline{\mathbb{C}}$ of the following functions:

   a) $\dfrac{1}{e^z - 1} - \dfrac{1}{z}$ b) $e^{z - \frac{1}{z}}$ c) $z(e^{\frac{1}{z}} - 1)$

   d) $e^{\tan \frac{1}{z}}$ e) $z^2 \sin \frac{z}{z+1}$.

5. Evaluate the integral
$$\int_0^\infty \frac{x^4}{x^6 + 1}\, dx.$$

6. Find all possible values of
$$\int_0^1 \frac{dz}{1 + z^2}$$
for all paths going from 0 to 1 such that the integral converges.

7. Let P be a harmonic polynomial of two real variables. Show that the conjugate harmonic function is also a polynomial.

8. Find the infinite product expansion for $e^z - 1$.