MATH 530 Qualifying Exam

August 1997

Notation: $D_1(0)$ denotes the unit disk, $\{z \in \mathbb{C} : |z| < 1\}$.

1. Give a careful statement and proof of *exactly* one of the following theorems:

- The Fundamental Theorem of Algebra
- The Partial Fractions Decomposition Theorem

2. Find an analytic function that maps $\Omega = D_1(0) - [0, 1]$ *one-to-one* and *onto* the vertical strip $\mathcal{S} = \{z \in \mathbb{C} : 0 < \text{Re } z < 1\}$. Is the mapping you found unique? Explain.

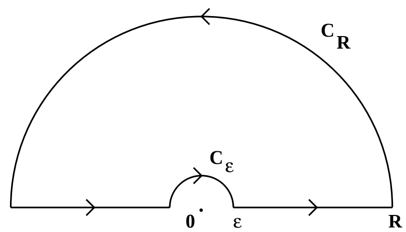
3. Prove the following mini-version of the Mittag-Leffler Theorem: There exists an analytic function $f(z)$ on $\mathbb{C} - \{n : n = 1, 2, 3, \ldots\}$ such that $f(z)$ has a simple pole at $z = n$ with principle part $1/(z - n)$ for each $n \in \mathbb{N}$. (You can use any theorems from Ahlfors *except* the Mittag-Leffler Theorem.)

4. Prove that an analytic function that maps the unit disk into itself must satisfy $|f'(0)| \leq 1$.

5. Evaluate

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

Hint: Integrate e^{iz}/z around the contour below. (Prove any limits you use).



6. Suppose that f is an analytic function on the complex plane minus the two points ± 1. Let γ_1 denote the curve given by $z_1(t) = 1 + e^{it}$ where $0 \leq t \leq 2\pi$ and let γ_2 denote the curve given by $z_2(t) = -1 + e^{it}$ where $0 \leq t \leq 2\pi$. Suppose that

$$\int_{\gamma_j} f(z)\, dz = 0$$

for $j = 1, 2$. First, explain why

$$\int_\gamma f(z)\, dz = 0$$

for any closed curve in $\mathbb{C} - \{\pm 1\}$. Next, prove from first principles that f has an analytic antiderivative on $\mathbb{C} - \{\pm 1\}$, i.e., show that there is an analytic function g on $\mathbb{C} - \{\pm 1\}$ such that $g' = f$.