MATH 530 Qualifying Exam

August 1995

Notation: $D_1(0)$ denotes the unit disk, $\{z \in \mathbb{C} : |z| < 1\}$.

1. A famous sequence of numbers is defined by $c_0 = 0$, $c_1 = 1$, and

$$c_n = c_{n-1} + c_{n-2} \qquad \text{for } n = 2, 3, 4 \ldots$$

Prove that the c_n are Taylor coefficients at the origin of the rational function, $z/(1 - z - z^2)$. What is the radius of convergence of the series?

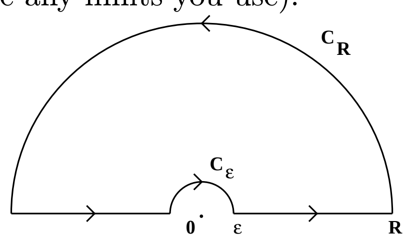
2. Find an analytic function that maps $\Omega = D_1(0) - [0, 1]$ *one-to-one* and *onto* the left half-plane $H = \{z \in \mathbb{C} : \text{Re } z < 0\}$. Is the mapping you found unique? Explain.

3. Suppose that f is a continuous function on $\{z \in \mathbb{C} : \text{Im } z \geq 0\}$ that is analytic on $\{z \in \mathbb{C} : \text{Im } z > 0\}$. Show that if f vanishes on a non-empty interval (a, b) on the real axis, then f must vanish identically. Is the same result true if the word "analytic" is replaced by the word "harmonic?" Explain.

4. Evaluate

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\, dx$$

where a and b are positive real constants. *Hint:* Integrate $\dfrac{e^{iaz} - e^{ibz}}{z^2}$ around the contour below. (Prove any limits you use).



5. Suppose that f is an entire function such that for every compact set $K \subset \mathbb{C}$, the inverse image $f^{-1}(K)$ is also compact. Prove that $f(\mathbb{C}) = \mathbb{C}$.

6. Suppose that f is a non-vanishing analytic function on the complex plane with the two points ± 1 deleted. Let γ_1 denote the curve given by $z_1(t) = 1 + e^{it}$ where $0 \leq t \leq 2\pi$ and let γ_2 denote the curve given by $z_2(t) = -1 + e^{it}$ where $0 \leq t \leq 2\pi$. Suppose that

$$\frac{1}{2\pi i} \int_{\gamma_j} \frac{f'(z)}{f(z)}\, dz$$

is divisible by 2 for $j = 1, 2$. First, explain why

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)}\, dz$$

must be divisible by 2 for any closed curve in $\mathbb{C} - \{\pm 1\}$. Next, prove that f has an analytic square root on $\mathbb{C} - \{\pm 1\}$, i.e., show that there is an analytic function g on $\mathbb{C} - \{\pm 1\}$ such that $g^2 = f$.