1. Show that the Cauchy problem [20pt]

$$\begin{aligned} &yu_x + xu_y = 0 && \text{in } U = \{(x,y) \in \mathbb{R}^2 : x > 0\}, \\ &u(x,0) = e^{-x^2} && \text{for any } x > 0, \end{aligned}$$

has a classical solution in all of U. Show, however, that the solution is not unique.

[*Hint.* Does every projected characteristic intersect $\Gamma = \{(x,0) : x > 0\}$?]

2. Let $\Omega_1 \subset \Omega_2$ be two bounded domains with smooth boundaries and functions $u_k \in C^2(\overline{\Omega}_k)$ and constants λ_k, $k = 1, 2$, are such that [20pt]

$$u_k > 0 \quad \text{in } \Omega_k, \qquad -\Delta u_k = \lambda_k u_k \quad \text{in } \Omega_k, \qquad u_k = 0 \quad \text{on } \partial\Omega_k, \qquad k = 1, 2.$$

Prove that

(a) $\lambda_k > 0$, $k = 1, 2$

(b) $\lambda_2 \leq \lambda_1$.

[*Hint.* Apply a Gauss-Green formula in an appropriate domain and use that $\partial u_k / \partial \nu$ has a sign on $\partial\Omega_k$. You have to justify the latter fact.]

3. Let u be a C^2 solution of the initial value problem [20pt]

$$\begin{aligned} u_{tt} - \Delta u = 0 \quad & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u = g(x), \quad u_t = h(x) \quad & \text{on } \mathbb{R}^3 \times \{0\}, \end{aligned}$$

where g, h are smooth functions with *compact support*. Prove that

$$E(t) := \int_{\mathbb{R}^3} |u(x,t)|^p dx \to 0 \quad \text{as } t \to \infty$$

for any $p > 2$.

[*Hint:* Identify the support of $u(\cdot,t)$ as well as prove a pointwise bound $|u(x,t)| \leq C/t$.]

4. Let u be a bounded solution of the initial value problem for the heat equation [20pt]

$$\begin{aligned} \Delta u - u_t &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) &= g \quad \text{on } \mathbb{R}^n, \end{aligned}$$

with $g \in C_0(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} g(y) dy > 0.$$

Prove that for any $R > 0$, there exists $T = T_R > 0$ such that

$$u(x, t) > 0 \quad \text{for all } |x| \leq R,\ t \geq T.$$

[*Hint:* Show that

$$(4\pi t)^{n/2} u(x, t) \to \int_{\mathbb{R}^n} g(y) dy \quad \text{as } t \to \infty,$$

uniformly for $|x| \leq R$.]

5. Consider the square $\Omega = (-1,1) \times (-1,1) \subset \mathbb{R}^2$ and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solve [20pt]

$$\Delta u = -1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

(a) Show that $u > 0$ in Ω

(b) Show that $u(x_2, x_1) = u(x_1, x_2)$ and $u(\pm x_1, \pm x_2) = u(x_1, x_2)$ for any $x = (x_1, x_2) \in \Omega$.

(c) Show that

$$u_{x_i} \leq 0 \quad \text{in } \overline{\Omega} \cap \{x_i \geq 0\}, \qquad i = 1, 2.$$

In particular,

$$u_{x_1}, u_{x_2} \leq 0 \quad \text{in } \overline{\Omega} \cap \{x_1 \geq 0, x_2 \geq 0\}.$$

[*Hint:* Since u_{x_i} are harmonic, it is enough to show that $u_{x_i} \leq 0$ on $\partial(\Omega \cap \{x_i > 0\})$; use both (a) and (b)]

(d) Use (b) and (c) to conclude that

$$u(0,0) = \sup_{\Omega} u.$$