MA 52300 QUALIFYING EXAMINATION
August 2018 (Professors Phillips and Wang)

1. (15 pt.) Let $u \in C^2(\mathbb{R}^n)$ such that

i) $u \geq 0 \qquad \text{on} \quad \mathbb{R}^n,$

ii) $\Delta u = 1 \qquad \text{on} \quad \mathbb{R}^n.$

Prove for each ball $B(p,r) \subset \mathbb{R}^n$ that $\displaystyle\sup_{x \in \overline{B}(p,r)} u(x) \geq \frac{r^2}{2n}$.

Problem 1 more space

2. (15 pt.) Let $f \in C_c^2(B(o,1))$ where $B(o,1) \subset \mathbb{R}^3$. Set $w(x) = \Gamma * f(x)$ where $\Gamma(x)$ is the fundamental solution for the Laplacian in $\mathbb{R}^3$.

a) Estimate $w(x)$ to prove that there is a $C < \infty$ so that

$$|w(x)| \leq \frac{C}{|x|} \qquad \text{for} \quad |x| \geq 2.$$

b) If in addition we have

$$\int_{B(o,1)} f(x)dx = 0$$

prove that there is $C_1 < \infty$ so that

$$|w(x)| \leq \frac{C_1}{|x|^2} \quad \text{for} \quad |x| \geq 2.$$

Problem 2 more space

3. (30 pt.) Consider the three problems:

i)

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \\ u(x,0) &= 1, \\ u_t(x,0) &= \sin x. \end{aligned}$$

ii)

$$\begin{aligned} u_t - u_{xx} &= 0, \\ u(x,0) &= 1, \\ u_t(x,0) &= \sin x. \end{aligned}$$

iii)

$$\begin{aligned} u_{tt} + u_{xx} &= 0, \\ u(x,0) &= 1, \\ u_t(x,0) &= \sin x. \end{aligned}$$

a) Determine for each of the three problems if there is a bounded solution $u(x,t) \in C^2(\mathbb{R}^2_+)$ where $\mathbb{R}^2_+ = \{(x,t) | \quad x \in \mathbb{R},\, t \geq 0\}$. Justify your answers.

Problem 3 continued.

b) Determine for each of the problems if there is a local solution $u(x,t) \in C^2(B_+(o,\varepsilon))$ for some $\varepsilon > 0$ where

$$B_+(o,\varepsilon) = \{(x,t) |\ x^2 + t^2 < \varepsilon^2, \quad t \geq 0\}.$$

4. (20 pt.)

a) Given $\tilde{f}(x,t), \tilde{g}(x), \tilde{h}(x)$ for $x \in \mathbb{R}$ and $t > 0$ write down the representation for the solution to

$$\begin{aligned} \tilde{u}_{tt} - \tilde{u}_{xx} &= \tilde{f}(x,t) \qquad x \in \mathbb{R}, \quad t > 0, \\ \tilde{u}(x,0) &= \tilde{g}(x) \qquad x \in \mathbb{R}, \\ \tilde{u}_t(x,0) &= \tilde{h}(x) \qquad x \in \mathbb{R}. \end{aligned}$$

Problem 4 continued.

b) Let

$$E = \{(x,t) | \; 0 \le x < \infty, \; 0 \le t < \infty\}.$$

Given

$$\begin{aligned} &f \in C^1(E), \\ &g \in C^2([0,\infty)), \\ &h \in C^1([0,\infty)) \end{aligned}$$

such that $g(0) = h(0) = 0$ and $f(0,t) = 0$ for $t \ge 0$ solve

$$\begin{aligned} &u_{tt} - u_{xx} = f \quad && \text{for} \quad \text{x} > 0 \quad \text{and} \quad \text{t} > 0, \\ &u(0,t) = 0 && \text{for} \quad t > 0, \\ &u(x,0) = g(x) && \text{for} \quad x \ge 0, \\ &u_t(x,0) = h(x) && \text{for} \quad x \ge 0 \end{aligned}$$

by extending f, g and h for $x < 0$ and using a).

5. (20 pt.) Solve

$$\begin{aligned} &2xyu_x + u_y = u^{1/2} \quad \text{for} \quad (x, y) \in U \subset \mathbb{R}^2, \\ &u(x, 0) = x^2 \quad \text{for} \quad x > 0. \end{aligned}$$

where U is a neighborhood of the positive x axis.

Problem 5 more space