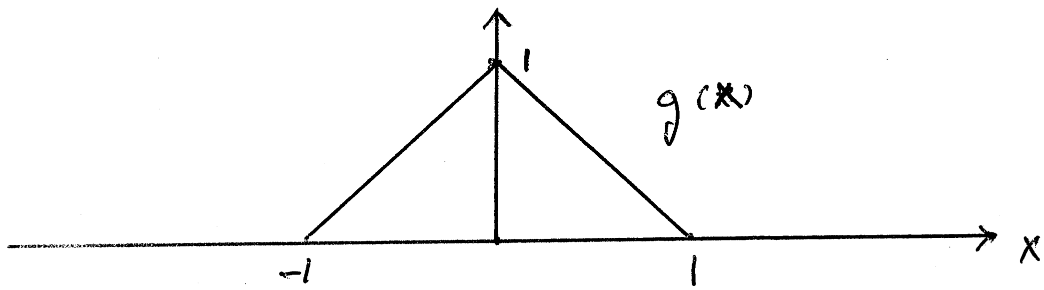
MATH 52300 PDE QUALIFYING EXAMINATION AUGUST 2015

1. **(30 pts)** Set $g(x) = \begin{cases} 1-|x| & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$



Let $u(x,t)$ denote the bounded solution for each of the three problems below. In each case express u in terms of g. Use the representation to explain why or why not $u(x,1) \in C^1(\mathbb{R})$.

a) $$\begin{aligned} &u_{tt} - u_{xx} = 0 && 0 < t,\ -\infty < x < \infty, \\ &u(x,0) = g(x) && -\infty < x < \infty, \\ &u_t(x,0) = 0 && -\infty < x < \infty. \end{aligned}$$

b) $$\begin{aligned} &u_t - u_{xx} = 0 && 0 < t,\ -\infty < x < \infty \\ &u(x,0) = g(x) && -\infty < x < \infty. \end{aligned}$$

c) $$\begin{aligned} &u_{tt} + u_{xx} = 0 && 0 < t,\ -\infty < x < \infty, \\ &u(x,0) = g(x) && -\infty < x < \infty. \end{aligned}$$

2. **(15 pts)** Let $u(x,t) \in C^2$ and solve

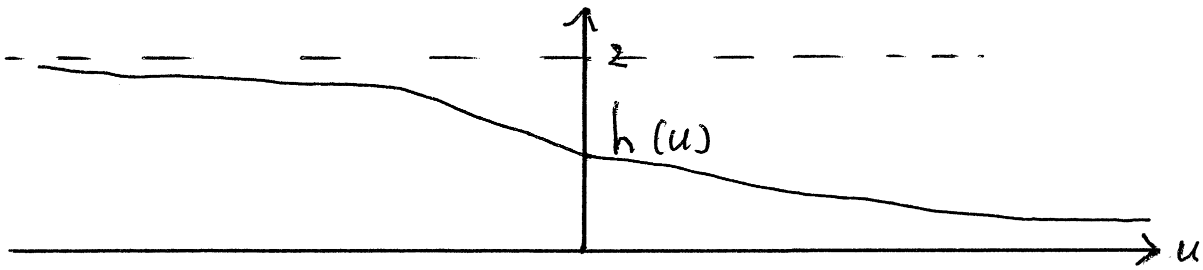
$$
\begin{aligned}
u_{tt} - u_{xx} + 3u_t + u &= 0 \qquad & 0 < x < \pi, t > 0 \\
u(0,t) = u(\pi,t) &= 0 & 0 < t, \\
u(x,0) = g(x),\ u_t(x,0) &= h(x), & 0 \le x \le \pi.
\end{aligned} \tag{2.1}
$$

a) Show that $\mathcal{E}(t) = \int_0^\pi [u_t^2 + u_x^2 + u^2]dx$ is nonincreasing.

b) Show that (2.1) has at most one solution.

c) Find the solution to (2.1) is the case when $g(x) = \sin x$ and $h(x) = 0$. (Hint; look for a separable solution.)

3. **(20 pts)** Let $h(u) \in C^1(\mathbb{R})$ such that $0 < h(u) < 2$ and $h'(u) < 0$ as pictured below



Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth $\partial\Omega$. Assume $\Omega \subset \overline{B}_1(0)$.

a) State the maximum principle for a function $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$ with $\Delta u \geq 0$ in Ω.

b) Set $u_0 = 0$ and let $u_1 \in C^2(\overline{\Omega})$ solve

$$\begin{aligned} \Delta u_1 &= h(u_0) = h(0) && \text{in } \Omega, \\ u_1 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Show that $u_1 \leq 0$ in Ω.

c) Let $u_2 \in C^2(\overline{\Omega})$ solve

$$\begin{aligned} \Delta u_2 &= h(u_1(x)) && \text{in } \Omega \\ u_2 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Show that $u_2 \leq u_1$ in Ω.

d) Find a function $v \subset C^2(\overline{\Omega})$ so that

$$\begin{aligned} \Delta v &\geq 2 && \text{in } \Omega \\ v &\leq 0 && \text{on } \overline{\Omega} \end{aligned}$$

and show that $v \leq u_2$ in Ω.

4. **(20 pts)** Let $f(x,t) \in C^2(\mathbb{R}^n \times [0,\infty))$ such that $f(x,t) = 0$ if $|x| \geq 4$. Let $u(x,t)$ solve

$$\begin{aligned} u_t - \Delta u &= f(x,t) \text{ for } (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) &= 0 \qquad\qquad \text{for } x \in \mathbb{R}^n. \end{aligned}$$

a) Use Duhamel's principle to derive a representation for a solution $u(x,t)$.

b) Use the representation to show:

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \leq \int_0^t \sup_{x\in\mathbb{R}^n} |f(x,\tau)| d\tau,$$

5. **(15 pts)** A bounded function $u(x,t)$ is a weak solution to

$$u_t + 2uu_x = 0 \qquad \text{in } \mathbb{R}^2 \tag{5.1}$$

if $\displaystyle\iint_{\mathbb{R}^2} (u\varphi_t + u^2\ \varphi_x)\ dxdt = 0$ for all $\varphi \in C_c^1(\mathbb{R}^2)$.

a) For $c \in \mathbb{R}$ define $v_c(x,t)$ by

$$\begin{aligned} v_c(x,t) &= 2 \qquad\qquad \text{if } x < ct, \\ &= 0 \qquad\qquad \text{if } x > ct. \end{aligned}$$

Determine c so that v_c is a weak solution to (5.1).

b) Use the method of characteristics to find a classical solution to

$$\begin{aligned} u_t + 2uu_x &= 0 \qquad\qquad x \in \mathbb{R},\ 0 < t < \overline{T}, \\ u(x,0) &= -x \qquad\qquad x \in \mathbb{R} \end{aligned}$$

for some $\overline{T}$. Find the maximum value of $\overline{T}$ possible.