Qualifying exam in Partial Differential Equations

August, 2011
Name..Total: 160 points

I solved problems (give four numbers here)...

Solve any FOUR of the following five problems. You HAVE to specify which problems you solved. Only four will be graded.

The total for the exam is 160 points, so you can get more than maximum if you dropped a "cheap" one.

Make sure you give a VERY detailed response.

Problem 1 – 30 points. (a) Solve the initial value problem:

$$u_{tt} - u_{xx} = \cos t, \quad u(0, x) = \sin x, \quad u_t(0, x) = \cos x, \quad x \in \mathbb{R}, \quad t \geq 0.$$

(b) Does the solution exhibit resonance? Explain.
(c) Suggest a modification of the external force (currently, $\cos t$) such that the answer to the question about resonance is different. Explain your suggestion.

Problem 2 – 30 points. Solve using the method of characteristics:

$$2xt\, u_x + u_t = u, \quad u(x, 0) = x, \quad x, t \in \mathbb{R}$$

Problem 3 – 50 points. Establish the Harnack inequality, that is, prove that for each connected open set $V \subset\subset U$ there exists a positive constant C depending only on V such that

$$\sup_V u \leq C \inf_V u,$$

for all nonnegative harmonic functions u in a set U.

Here the notation $V \subset\subset U$ signifies that $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact.

Hint: Show first that whenever u is harmonic in a ball $B(x, 3r)$, $x \in U$, $r > 0$, we have for every $y \in B(x, r)$

$$C_1 u(y) \leq u(x) \leq C_2 u(y),$$

with constants C_1 and C_2 independent of u. Then argue using a suitable covering of V.

Problem 4 – 50 points[1]. This is a problem about Fourier transform. Depending on the normalization you use, your constants might be different. That's fine. Just let me know which definition you are using.

Consider the Cauchy problem for the generalized heat equation

$$u_t + (-\Delta)^{\alpha/2} u = 0, \quad u|_{t=0} = f, \quad f \in L^2(\mathbb{R}^n).$$

[1]A general idea of the problem is taken from: K. Yu. Osipenko, Extremal Problems for the Generalized Heat Equation and Optimal Recovery of its Solution from Inaccurate Data, Optimization Vol. 00, No. 00, January 2009

The fractional Laplacian operator $(-\Delta)^{\alpha/2}$ is defined as

$$(-\Delta)^{\alpha/2} g(x) = \mathcal{F}^{-1}(|\xi|^{\alpha} \mathcal{F} g(\xi))(x),$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is its inverse, and the boundary data is taken in the sense $\|u(t,\cdot) - f\|_{L^2(\mathbb{R}^n)} \to 0$ as $t \to 0$.

(a) Show that

$$u(t,x) = \mathcal{F}^{-1}(e^{-|\xi|^{\alpha} t} \mathcal{F} f(\xi))(x),$$

provides the solution to the Cauchy problem above. Make sure that you verify convergence to the boundary data in L^2, as stated above.

(b) Explain why it follows that

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-2|\xi|^{\alpha} t} \mathcal{F} f(\xi))^2 \, d\xi.$$

Furthermore, using this formula, prove that for any $0 \leq t_1 < \tau < t_2$

$$\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|u(t_1,\cdot)\|^{\frac{t_2-\tau}{t_2-t_1}}_{L^2(\mathbb{R}^n)} \|u(t_2,\cdot)\|^{\frac{\tau-t_1}{t_2-t_1}}_{L^2(\mathbb{R}^n)},$$

i.e., that $\log \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ is a convex function of t.

(c) Using Part (b), establish the "uniqueness backwards in time" for the generalized heat equation, i.e., show that if u_1, u_2 both solve

$$u_t + (-\Delta)^{\alpha} u = 0 \text{ in } \mathbb{R}^n \times (0,T]$$

in the sense described above (note that we do not prescribe $u_1(x,0), u_2(x,0)$!), and

$$u_1(x,T) = u_2(x,T) \quad \text{for all} \quad x \in \mathbb{R}^n,$$

then u_1 and u_2 are identically equal for all $t \in (0,T]$, $x \in \mathbb{R}^n$.

Problem 5 – 50 points. Let $a, c > 0$ be positive constants. The telegrapher's equation

$$u_{tt} + a u_t = c^2 u_{xx}, \quad x \in (0,1), \quad t > 0,$$

represents a damped version of the wave equation. Consider the Dirichlet boundary value problem with

$$\textit{boundary conditions}: \quad u(t,0) = u(t,1) = 0$$

and

$$\textit{initial conditions}: \quad u(0,x) = f(x), \quad u_t(0,x) = 0$$

(a) Find all separable solutions to the telegrapher's equation that satisfy the *boundary conditions*.

(b) Write down a series solution for the *initial* boundary value problem (i.e., satisfying both boundary and initial conditions). You do not have to verify orthonormality of the elements of a standard Fourier series, the fact that they form a basis etc. Just write the formula for u in a series form and for the coefficients of the series.