Math 523
Qualifying Examination
January 2006
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Name...

I. D. no. ....

Problem	Score	Max. pts.
1		20
2		20
3		20
4		20
5		20
6		20
Total		120

Problem 1. Let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ denote the unit sphere centered at the origin, i.e. $\mathbb{S}^{n-1} = \{\omega \in \mathbb{R}^n \mid |\omega| = 1\}$, and consider the function $\phi \in C^\infty(\mathbb{R}^n)$ defined as follows

$$\phi(x) \ = \ \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}\ <x,\omega>}\ d\sigma(\omega)\ , \qquad \lambda > 0\ , \quad x \in \mathbb{R}^n\ ,$$

where $d\sigma$ denotes the $(n-1)$-dimensional surface measure on $\mathbb{S}^{n-1}$, and $i^2 = -1$. Prove that the function $u \in C^\infty(\mathbb{R}^{n+1})$ defined by

$$u(x,t) \ = \ e^{-\lambda t}\ \phi(x)\ ,$$

solves the heat equation $Hu = \Delta u - u_t = 0$ in $\mathbb{R}^{n+1}$.

Problem 2. Consider in the plane the solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad Q \,, \\ u = \phi_1 & \text{on} \quad \partial Q \,, \end{cases}$$

where $Q = \{(x, y) \in \mathbb{R}^2 \mid |x| < 1, |y| < 1\}$, and ϕ_1 is the function which equals 1 on one of the edges of the square Q, and 0 on the remaining three edges (never mind the fact that such a ϕ_1 is not continuous on ∂Q, you can still uniquely solve the Dirichlet problem with such boundary datum).

a) Compute $u(0)$.

b) Can you tell what is $u(0)$ for the solution of an analogous Dirichlet problem for a regular polygon with n edges, and with boundary datum 1 on one edge and 0 anywhere else?

Hint: Use the invariance of Laplace's operator with respect to orthogonal transformations and the maximum principle...

Problem 3. Let $n \geq 2$ and $u \in C^2(\mathbb{R}^n)$ be a solution in $\mathbb{R}^n$ of the equation $\Delta u = |x|^k$, for some $k \geq 0$. With σ_{n-1} being the $(n-1)$-dimensional measure of the unit sphere, denote by

$$M_u(r) \;=\; \frac{1}{\sigma_{n-1} r^{n-1}} \int_{\partial B(0,r)} u(y) d\sigma(y) \ ,$$

the spherical mean of u over the sphere centered at the origin with radius r.
a) Prove that for every $r \geq 0$ one has

$$M_u(r) \;=\; u(0) \;+\; \frac{r^{k+2}}{(n+k)(k+2)}$$

b) Show that an estimate such as

$$|u(x)| \;\leq\; C\ (1 + |x|^{k+2-\epsilon}) \ , \qquad\qquad x \in \mathbb{R}^n \ ,$$

is impossible for $C \geq 0$ and $\epsilon > 0$.

Problem 4. **Problem 4.** A C^1 open set $A \subset \mathbb{R}^n$ is called *starlike* if, denoted by ν the outer unit normal to ∂A, one has

$$< x, \nu(x) > \ \geq \ 0 \qquad\qquad \text{for every} \quad x \in \partial A \ .$$

When the inequality is strict everywhere on ∂A, then A is said *strictly starlike*.

Let $\Omega_1 \subset \overline{\Omega}_1 \subset \Omega_0$ be two connected, bounded, starlike domains, and suppose that the problem

$$\begin{cases} \Delta u \ = \ 0 & \qquad \text{in} \quad \Omega = \Omega_0 \setminus \overline{\Omega}_1 \ , \\ u \ = \ 1 \qquad \text{on} \quad \partial\Omega_1 \ , \ u \ = \ 0 & \qquad \text{on} \quad \partial\Omega_0 \ , \end{cases} \tag{1}$$

admits a solution $u \in C^3(\overline{\Omega})$.

(i) Show that $0 < u < 1$ in Ω.

(ii) Compute the equation satisfied by $v(x) \stackrel{def}{=} < x, \nabla u(x) >$ in Ω, and use it to prove that every C^1 level set $E_t = \{x \in \Omega \mid u(x) > t\}$ of u, $0 < t < 1$, is strictly starlike.

Problem 5. Let u be a solution of the initial-value problem for the wave equation

$$\begin{cases} u_{tt} \;-\; \Delta u \;=\; 0 & \text{in} \quad \mathbb{R}^3 \times [0,\infty)\ , \\ u(x,0) \;=\; \phi\ , & u_t(x,0) \;=\; \psi(x)\ , \end{cases}$$

where $\phi, \psi \in C^\infty(\mathbb{R}^3)$. For every $t \geq 0$ let

$$E(t) \;=\; \sum_{|\alpha|+j\leq 2} \int_{\mathbb{R}^3} |D_x^\alpha D_t^j u(x,t)|\ dx\ .$$

Use Kirchoff's formula and an integration by parts that converts surface in solid integrals, to prove that if $E(0) < \infty$, then one has for every $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, with $t \geq 1$,

$$|u(x,t)| \;\leq\; \frac{C}{t}\ E(0)\ ,$$

for some constant $C \geq 0$ independent of u.

Hint: On $\partial B(x,t)$ one has $1 \equiv < \frac{y-x}{t}, \frac{y-x}{t} > = < \frac{y-x}{t}, \nu >$, where ν is the outer unit normal on $\partial B(x,t)$ ...

Problem 6. Let u be the solution of the initial-value problem

$$\begin{cases} \Delta u \ - \ u_t \ = \ 0 \ , & \text{in} \quad \mathbb{R}^n \times (0, \infty) \ , \\ u(x,0) \ = \ \phi(x) \ , & x \in \mathbb{R}^n \ , \end{cases}$$

where $\phi \in C(\mathbb{R}^n)$ is a function such that for a fixed $R > 0$, $\phi(x) = 0$ for $|x| > R$. Prove that there exists a $C = C(n) > 0$, such that for every $x \in \mathbb{R}^n$ and $t > 0$

$$|\nabla_x u(x,t)| \ \leq \ \frac{C}{\sqrt{t}} \left(\max_{|y| \leq \ R} |\phi(y)| \right) \ .$$

(Recall that

$$G(x,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} \ \exp\left(-\frac{|x|^2}{4t}\right) \ , & t > 0 \ , \\ 0 \ , & t \leq 0 \ , \end{cases}$$

is the fundamental solution of the heat equation $Hu = \Delta u - u_t = 0$ in R^{n+1}, with singularity at $(0,0)$)