QUALIFYING EXAMINATION
AUGUST 2003
MATH 523 - Prof. Tournier

Each problem is worth 20 points.

1. Find the solution of the problem

$$u_y + u_x^2 z = 0$$
$$u(x,0) = x^2$$

2. For $x^2 + y^2 < 1$, let

$$v(x,y) = \frac{1-(x^2+y^2)}{2\pi} \int_0^{2\pi} \frac{\cos\theta \sin\theta}{(x-\cos\theta)^2 + (y-\sin\theta)^2}\, d\theta$$

Find the explitic formula for v and justify your answer.

3. Let $B_1(0)$ be the unit ball in $\mathbb{R}^3$. Suppose $u \in C(\overline{B_1(0)}) \cap C^2(B_1(0)\backslash\{0\})$ and $\Delta u = 0$ in $B_1(0)\backslash\{0\}$.

   a) Show that $\int_{B_1(0)} u(x)\Delta\phi(x)dx = 0$ for all $\phi \in C_0^\infty(B_1(0))$.

   b) Give an example of a function $u \in C^2(\overline{B_1(0)}\backslash\{0\})$ such that $\Delta u = 0$ in $B_1(0)\backslash\{0\}$ and such that the result in a) does not hold. Compute $\int_{B_1(0)} u(x)\Delta\phi(x)dx$ for $\phi \in C_0^\infty(B_1(0))$.

4. Let u be bounded and continuous on $\mathbb{R}^n \times [0,T]$ and $u_t - \Delta u = 0$ on $\mathbb{R}^n \times (0,T)$.

   a) Show that $\sup\limits_{\mathbb{R}^n \times [0,T]} u = \sup\limits_{x \in \mathbb{R}^n} u(x,0)$.

      Hint: Consider $v(x,t) = u(x,t) - \varepsilon(nt + |x|^2)$.

   b) Write a formula for the solution u of

$$u_t - \Delta u = 0$$
$$u(x,0) = g(x) \qquad \text{where } g \text{ is continuous}$$

      where g is continuous with compact support in $\mathbb{R}^n$.

5. Find a formula for the solution of

$$u_{tt} - u_{xx} + u = 0 \qquad \text{in} \quad \mathbb{R} \times (0, \infty)$$

such that

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

where $f, g \in C_0^\infty(\mathbb{R})$.
Hint: Let $v(x,y,t) = h(y)u(x,t)$.
Find h so that v solves the 2–dimensional wave equation.