QUALIFYING EXAMINATION

JANUARY 2001

MATH 523 – Prof. Cai

1. (20pts) Solve the initial value problem

$$\begin{cases} xu\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} &= 0, \\ u(x,0) &= x. \end{cases}$$

2. (20pts) For $n = 2$ dimensions, let

$$v = \frac{1}{8\pi} r^2 \log r, \quad r = |x - \xi|, \quad x, \xi \in R^2.$$

(a) Show that $\Delta^2 v = 0$ for $r > 0$.

$$\text{[Hint:} \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad \text{and} \quad \Delta v = \frac{1 + \log r}{2\pi}.\text{]}$$

(b) Show that for $u \in C^4(\Omega)$ and $\xi \in \Omega$

$$u(\xi) = \int_\Omega v\Delta^2 u\, dx - \int_{\partial\Omega} \left(v\frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial v}{\partial n} + \Delta v \frac{\partial u}{\partial n} - u\frac{\partial \Delta v}{\partial n} \right) dS.$$

[Hint: Use Green's identity $\int_{\Omega_\rho} v\Delta u\, dx = \int_{\Omega_\rho} u\Delta v\, dx + \int_{\partial\Omega_\rho} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) dS$ with u replaced by Δu where $\Omega_\rho = \Omega \setminus B(\xi, \rho)$ and $B(\xi, \rho)$ is a disk centered at ξ with radius ρ.]

(c) Show that v is a fundamental solution for the operator Δ^2.

3. (20pts) Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ and let n be the outward unit vector normal to the boundary. Prove that the following problem has unique solution in $C^2(\Omega) \cap C^0(\bar{\Omega})$:

$$\begin{cases} \Delta u &= f \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} + \beta u &= g \quad \text{on} \quad \partial\Omega \end{cases}$$

where $\frac{\partial u}{\partial n}$ stands for the outward normal derivative, β is a positive number, and f and g are sufficient smooth.

4. (20pts) Use Hadamard's method of descent to derive an explicit formula of the solution for two-dimentional wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u &= 0 \quad \text{in} \quad R^2 \times R_+, \\ u(x,0) &= f(x) \quad \text{in} \quad R^2 \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \quad \text{in} \quad R^2. \end{cases}$$

[Hint: The explicit formula for three-dimensional wave equation is

$$u(x,t) = \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} g(y)\, dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t} \int_{|y-x|=ct} f(y)\, dS_y\right).]$$

5. (20pts) Let ω be an open bounded set of R^n, for a fixed $T > 0$ let

$$\Omega = \{(x,t) \,|\, x \in \omega,\ 0 < t < T\},$$

and denote the "lower" boundary by

$$\partial'\Omega = \{(x,t) \,|\, \text{either } x \in \partial\omega,\ 0 \le t \le T \text{ or } x \in \omega,\ t = 0\}.$$

Assume that $u \in C^0(\bar{\Omega})$, $u_t \in C^0(\Omega)$, and $u_{x_i x_k} \in C^0(\Omega)$ for $i,k = 1, ..., n$, and $\frac{\partial u}{\partial t} - \Delta u \le 0$. Prove that

$$\max_{\bar{\Omega}} u = \max_{\partial'\Omega} u.$$