QUALIFYING EXAMINATION

JANUARY 1999

MATH 523 - Prof. Zachmanoglou

(15) 1. Solve the initial value problem

$$3z_x + z_y = z^3$$
$$z(x,0) = \frac{1}{\sqrt{x}}, \quad 0 < x.$$

(15) 2. Let Ω be a bounded normal domain in $\mathbb{R}^n$ with smooth boundary S, and let $\vec{n}$ be the exterior unit normal on S. Prove uniqueness of solution of the Neumann problem

$$\nabla^2 u - u = f, \qquad x \in \Omega$$
$$\frac{\partial u}{\partial n} = g, \qquad x \in S$$

assuming that f, g and u are sufficiently smooth.

(20) 3. (a) State carefully Liouville's theorem for harmonic functions in $\mathbb{R}^n$.

(b) Prove Liouville's theorem for $n = 2$, using the Poisson integral formula for the solution of the Dirichlet problem in a circle of radius a centered at the origin and with Dirichlet data f,

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \varphi)} f(\varphi) d\varphi, \quad r < a.$$

(10) 4. (a) Consider the wave equation in one space variable

$$u_{xx} - u_{tt} = 0.$$

Use a change of coordinates and integration to obtain its general solution

$$u(x,t) = F(x+t) + G(x-t),$$

where F and G are arbitrary C^2 functions of one variable.

(b) Consider the initial value problem

$$\begin{aligned} u_{xx} - u_{tt} &= 0; && -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) &= \varphi(x); && -\infty < x < \infty \\ u_t(x,0) &= \psi(x); && -\infty < x < \infty \end{aligned}$$

where φ and ψ are given functions in $C^2(\mathbb{R})$. Use the general solution in (a) to obtain the formula for the solution of the problem.

(20) 5. Consider the initial-boundary value problem for the heat equation in one space variable,

$$\begin{aligned} u_t - u_{xx} &= 0; && 0 < x < L,\ 0 < t, \\ u(x,0) &= \varphi(x); && 0 \le x \le L, \\ u(0,t) = 0, u(L,t) &= 1; && 0 \le t, \end{aligned}$$

where φ is continuous and has a sectionally continuous derivative on $[0, L]$, and $\varphi(0) = 0$, $\varphi(L) = 1$.

(a) Find the solution $u(x,t)$ of the problem. Make sure to give the formula for the coefficients of your series.

(b) Is $u(x,t)$ of class C^∞ when $t > 0$. Answer yes or no.

(c) Find $\lim_{t\to\infty} u(x,t)$.

(10) 6. (a) Give the form of a general linear partial differential operator $P(x, D)$ of order m in $\mathbb{R}^n$. Explain the notation.

(b) Let F be a C^1 function of n variables with nonvanishing gradient, and suppose that the level surfaces of F are characteristic surfaces of $P(x, D)$. Give the partial differential equation that F must satisfy, state the order of this equation, and whether the equation is linear or nonlinear.

(10) 7. (a) Write down Kirchhoff's formula for the solution of the initial value problem for the wave equation in three space variables,

$$\begin{aligned} u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} &= 0; && x \in \mathbb{R}^3, \quad 0 \le t \\ u(x,0) = 0,\ u_t(x,0) &= p(x); && x \in \mathbb{R}^3 \end{aligned}$$

where $p \in C^2(\mathbb{R}^3)$.

(b) Prove that your solution in (a) satisfies

$$\lim_{t\to 0^+} u(x,t) = 0.$$