QUALIFYING EXAMINATION
JANUARY 1998
MATH 523

(17 pts) 1. a) Find a solution of

$$(*) \qquad \begin{cases} xu_y - yu_x = u + 1 \\ \text{with } u(x,x) = x^2 \end{cases}$$

that is valid for all (x, y) in some neighborhood of $(1, 1)$.

b) What is the largest domain of definition for the solution obtained in 1.a)?

c) Identify all points (x_0, x_0) such that there exists a unique solution of $(*)$ in some neighborhood of (x_0, x_0). Explain your reasoning.

(16 pts) 2. Let D be a smooth bounded domain in $\mathbb{R}^n$ and let ν be the outward unit normal on ∂D. Prove that the problem

$$\triangle u = f \text{ in } D,$$
$$\frac{du}{d\nu} = 1 \text{ on } \partial D$$

has no C^2 solution for some functions f in $C(\overline{D})$. (Hint: Use the divergence theorem.)

(17 pts) 3. Consider the problem:

$$(**) \qquad \begin{aligned} u_{xx} + x \cdot u_{xy} + (2y - x^2)u_{yy} - u_y = 0, \\ u(x, 0) = x^2, \quad u_y(x, 0) = x. \end{aligned}$$

a) Determine the regions in the xy plane in which the first equation in $(**)$ is hyperbolic, parabolic, or elliptic.

b) Let A denote the x axis, $A = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. Find all points P_0 in A such that there exists a unique real analytic solution of $(**)$ in a sufficiently small neighborhood of P_0 in $\mathbb{R}^2$. State carefully a theorem which justifies your answer.

c) Let u be a real analytic solution of $(**)$ in some neighborhood of $(1, 0)$. Is $u_{xyy}(1, 0)$ uniquely determined? If so, compute its value.

(16 pts) 4. a) Let $\vec{x} = (x_1, x_2, x_3)$ and $B_2 = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| < 2\}$. Find a formula (in terms of explicit functions or integrals of explicit functions on appropriate domains) for the solution in $\overline{B}_2$ of

$$(\triangle u)(x_1, x_2, x_3) = \begin{cases} 1, & \text{for } |(x_1, x_2, x_3)| \leq 1 \\ 0, & \text{for } 1 < |(x_1, x_2, x_3)| \leq 2 \end{cases}$$

and $u(x_1, x_2, x_3) = 4$ when $|(x_1, x_2, x_3)| = 2$.
(<u>Hint</u>: The Green's function for the ball, $B_R = \{\vec{y} \in \mathbb{R}^3 : |\vec{y}| < R\}$, is given by:

$$G(\vec{y}, \vec{\xi}) = \frac{1}{4\pi|\vec{y} - \vec{\xi}|} - \frac{R}{4\pi|\vec{\xi}| \cdot |\vec{y} - \vec{\xi}*|}$$

for $(\vec{y}, \vec{\xi})$ in B_R where $\vec{\xi}* = \frac{R^2}{|\vec{\xi}|^2}\vec{\xi}$.

b) What can you say about the maximum or minimum value of u on $\overline{B}_2$?

(17 pts) 5. a) Find a solution of:

$$(***) \qquad u_t - u_{xx} = x \text{ for } 0 < x < 1, \quad t > 0$$

with initial and boundary conditions

$$u(x, 0) = 0 \quad \text{for} \quad 0 \leq x \leq 1,$$
$$u(0, t) = u(1, t) = 0 \quad \text{for} \quad t > 0.$$

(Hint: Use separation of variables and Fourier series after subtracting a convenient time-independent solution of $(***)$. You may express the Fourier coefficients as integrals of explicit functions on appropriate domains without evaluating these integrals.)

b) State and prove a uniqueness result for this problem.

(17 pts) 6. Let $\vec{x} = (x_1, x_2, x_3)$. Recall that the solution of $u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0$ for $(x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$ satisfying $u(\vec{x}, 0) = 0, u_t(\vec{x}, 0) = g(\vec{x})$ for $\vec{x} \in \mathbb{R}^3$ is given by:

$$u(\vec{x}, t) = \frac{1}{4\pi t} \int_{\{\vec{y} \in \mathbb{R}^3 : |\vec{x} - \vec{y}| = t\}} g(y) ds_y = \frac{t}{4\pi} \int_{\{\vec{\xi} \in \mathbb{R}^3 : |\vec{\xi}| = 1\}} g(x + t\xi) ds$$

where ds_y represents surface measure on $\{\vec{y} \in \mathbb{R}^3 : |\vec{y} - \vec{x}| = t\}$ and ds represents surface measure on $\{\vec{\xi} \in \mathbb{R}^3 : |\xi| = 1\}$. Assume

$$g(\vec{x}) = \begin{cases} \cos(|\vec{x}|^2), & \text{for } |\vec{x}| \leq \sqrt{\frac{\pi}{2}} \\ 0, & \text{for } |\vec{x}| > \sqrt{\frac{\pi}{2}}. \end{cases}$$

a) Identify $\{(\vec{x}, t) \in \mathbb{R}^3 \times (0, \infty) : u(\vec{x}, t) = 0\}$.

b) If $|\vec{x}| = \sqrt{\pi}$, what is the first time t_0 at which $u(\vec{x}, t_0)$ is nonzero?

c) Is the solution $u(\vec{x}, t)$ twice continuously differentiable on $\mathbb{R}^3 \times (0, \infty)$? Justify your answer.

d) Evaluate $\lim_{t \to \infty} u(\vec{x}, t)$.