MA 519: Introduction to Probability Theory
Qualifying Examination – January 2022

Your PUID: Scores: (1) (2) (3) (4) (5) (Total)

This examination consists of five questions, 20 points each, totaling 100 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them. Some formula sheets are attached at the end of the exam for your convenience. You can directly quote any formula from those sheets.

1. Let X and Y be two independent discrete random variables. For each of the following cases, compute the conditional distribution of X given $X+Y$, i.e. find

$$P\Big(X = i \Big| X + Y = j\Big)$$

Also, relate the conditional distribution to some common, i.e. well known, distribution.

(a) X is Poisson with parameter λ and Y is Poisson with parameter μ.

(b) X is Binomial with parameter n and p and Y is Binomial with parameter m and p.

(c) X and Y are Geometric with parameter p.

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2. Ten balls are to be distributed among 5 urns, with each ball independently going into urn i with probability p_i. Let X_i be the number of balls that go into urn i. Note that we have $\sum_{i=1}^{5} p_i = 1$ and $\sum_{i=1}^{5} X_i = 10$.

   (a) Find the joint distribution of $X_i, X_2, \ldots X_5$, i.e. find

   $$P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4, X_5 = x_5).$$

   (b) Find the (marginal) distribution of X_1.

   (c) Find the distribution of $X_1 + X_2$.

   (d) Find the distribution $X_1 + X_2 + X_3$.

   (e) Find $P(X_1 + X_2 = 4 | X_1 + X_2 + X_3 = 7)$.

   (Hint: You can do Parts (b)–(e) without knowing Part (a).)

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3. Let $X_1, X_2, \ldots, X_n$ be a collection of iid exponential random variables with parameter λ. Let

$$\begin{aligned}
Y_1 &= X_1, \\
Y_2 &= X_1 + X_2, \\
Y_3 &= X_1 + X_2 + X_3, \\
\cdot\cdot &= \cdots\cdots, \\
Y_n &= X_1 + \cdots + X_n.
\end{aligned}$$

(a) Find the joint pdf $p(y_1, y_2, \ldots y_n)$ of $Y_1, Y_2, \ldots, Y_n$.

(b) Find the conditional probability distribution of $Y_1, Y_2, \ldots Y_{n-1}$ given $Y_n = y_n$, i.e. find

$$p(y_1, y_2, \ldots, y_{n-1} \big| y_n).$$

(Hint: what is the marginal of Y_n?)

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4. Let X and Y be two independent, Gamma distributed random variables with parameter (α, λ) and (β, λ), respectively. Let

$$U = X + Y, \text{ and } V = \frac{X}{X+Y}.$$

   (a) Find the joint pdf between U and V.

   (b) Find the region in the uv-plane taken by U and V. (Note that the region in the xy-plane taken by X and Y is simply the first quadrant $\{x \geq 0, y \geq 0\}$.)

   (c) Find the marginals of U and V.

   (d) Are U and V independent?

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5. Let $T_1, T_2, \ldots$ be a sequence of i.i.d. exponentially distributed random variables with parameter λ, i.e. the p.d.f. of T_1 equals $\lambda e^{-\lambda t}$ for $t \geq 0$, or in terms of its c.d.f.,

$$P(T_1 \leq t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

Let $M_n = \max_{1 \leq k \leq n} T_k$. Compute the following limits as a function of x:

(a) $\lim_{n\to\infty} P(M_n \leq x)$

(b) $\lim_{n\to\infty} P\left(\frac{M_n}{\log n} \leq x\right)$

(c) $\lim_{n\to\infty} P\left(M_n - \frac{1}{\lambda}\log n \leq x\right)$

Explain in words (in four or five sentences) what the above results ((a), (b), and (c)) reveal about the behavior of M_n for large n, in particular the *relationship between* the results (a), (b), and (c).

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Common Discrete Distributions

- **Bernoulli**(p) X indicates whether a trial that results in a success with probability p is a success or not.

$$P\{X = 1\} = p$$
$$P\{X = 0\} = 1 - p$$

  $E[X] = p, \quad \mathrm{Var}(X) = p(1 - p).$

- **Binomial**(n,p) X represents the number of successes in n independent trials when each trial is a success with probability p.

$$P\{X = i\} = \binom{n}{i} p^i (1 - p)^{n-i}, \quad i = 0, 1, \ldots, n$$

  $E[X] = np, \quad \mathrm{Var}(X) = np(1 - p).$
  Note. Binomial$(1,p)$ = Bernoulli(p).

- **Geometric**(p) X is the number of trials needed to obtain a success when each trial is independently a success with probability p.

$$P(X = i) = p(1 - p)^{i-1}, \quad i = 1, 2, \ldots,$$

  $E[X] = \frac{1}{p}, \quad \mathrm{Var}(X) = \frac{1-p}{p^2}.$

- **Negative Binomial**(r,p) X is the number of trials needed to obtain a total of r successes when each trial is independently a success with probability p.

$$P(X = i) = \binom{i - 1}{r - 1} p^r (1 - p)^{i-r}, \quad i = r, r + 1, r + 2, \ldots$$

  $E[X] = \frac{r}{p}, \quad \mathrm{Var}(X) = r\frac{1-p}{p^2}.$
  Notes.
  1. Negative Binomial$(1,p)$ = Geometric(p).
  2. Sum of r independent Geometric(p) random variables is Negative Binomial(r,p)

- **Poisson**(λ) X is used to model the number of events that occur when these events are either independent or weakly dependent and each has a small probability of occurrence.

$$P\{X = i\} = e^{-\lambda}\lambda^i / i!, \quad i = 0, 1, 2, \ldots$$

  $E[X] = \lambda, \quad \mathrm{Var}(X) = \lambda.$
  Notes.
  1. A Poisson random variable X with parameter $\lambda = np$ provides a good approximation to a Binomial(n,p) random variable when n is large and p is small.
  2. If events are occurring one at a time in a random manner for which (a) the number of events that occur in disjoint time intervals is independent and (b) the probability of an event occurring in any small time interval is approximately λ times the length of the interval, then the number of events in an interval of length t will be a Poisson(λt) random variable.

- **Hypergeometric** X is the number of white balls in a random sample of n balls chosen without replacement from an urn of N balls of which m are white.

$$P\{X = i\} = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}, \quad i = 0, 1, 2, \ldots$$

The preceding uses the convention that $\binom{r}{j} = 0$ if either $j < 0$ or $j > r$.

With $p = m/N$, $E[X] = np$, $\quad \text{Var}(X) = \frac{N-n}{N-1}np(1 - p)$

Note. If each ball were replaced before the next selection, then X would be a Binomial(n, p) random variable.

- **Negative Hypergeometric** X is the number of balls that need be removed from an urn that contains $n + m$ balls, of which n are white, until a total of r white balls has been removed, where $r \leq n$.

$$P\{X = k\} = \frac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}} \frac{n - r + 1}{n + m - k + 1}, \quad k \geq r$$

$E[X] = r\,\frac{n+m+1}{n+1}$, $\quad \text{Var}(X) = \frac{mr(n+1-r)(n+m+1)}{(n+1)^2(n+2)}$

Common Continuous Distributions

- **Uniform** (a, b) X is equally likely to be near each value in the interval (a, b). Its density function is

$$f(x) = \frac{1}{b - a}, \quad a < x < b$$

$E[X] = \frac{a+b}{2}$, $\quad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$.

- **Normal**(μ, σ^2) X is a random fluctuation arising from many causes. Its density function is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

$E[X] = \mu$, $\quad \mathrm{Var}(X) = \sigma^2$
When $\mu = 0, \sigma = 1$, X is called a standard normal.
Notes.
1. If X is Normal(μ, σ^2), then $Z = \frac{X-\mu}{\sigma}$ is standard normal.
2. Sum of independent normal random variables is also normal.
3. An important result is the *central limit theorem*, which states that the distribution of the sum of the first n of a sequence of independent and identically distributed random variables becomes normal as n goes to infinity, for any distribution of these random variables that has a finite mean and variance.

- **Exponential**(λ) X is the waiting time until an event occurs when events are always occurring at a random rate $\lambda > 0$. Its density is

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$E[X] = \frac{1}{\lambda}$, $\quad \mathrm{Var}(X) = \frac{1}{\lambda^2}$, $P(X > x) = e^{-\lambda x}$, $x > 0$.
Note. X is memoryless, in that the remaining life of an item whose life distribution is Exponential(λ) is also Exponential(λ), no matter what the current age of the item is.

- **Gamma**(α, λ) When $\alpha = n$, X is the waiting time until n events occur when events are always occurring at a random rate $\lambda > 0$. Its density is

$$f(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0$$

where $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$ is called the gamma function.
$E[X] = \frac{\alpha}{\lambda}$, $\quad \mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$.
Notes.
1. Gamma$(1, \lambda)$ is exponential(λ).
2. If the random variables are independent, then the sum of a Gamma(α_1, λ) and a Gamma(α_2, λ) is a Gamma$(\alpha_1 + \alpha_2, \lambda)$.
3. The sum of n independent and identically distributed exponentials with parameter λ is a Gamma(n, λ) random variable.

- **Beta**(a, b) X is the distribution of a random variable taking on values in the interval $(0, 1)$. Its density is

$$f(x) = \frac{1}{B(a, b)} x^{a-1}(1 - x)^{b-1}, \quad 0 < x < 1$$

where $B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} dx$ is called the beta function.
$E[X] = \frac{a}{a+b}$ $\quad \mathrm{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$

Notes.
1. Beta(1, 1) and Uniform(0, 1) are identical.
2. The j^{th} smallest of n independent uniform (0, 1) random variables is a Beta($j, n - j + 1$) random variable.

- **Chi-Squared**(n) X is the sum of the squares of n independent standard normal random variables. Its density is

$$f(x) = \frac{e^{-x/2} x^{\frac{n}{2}-1}}{2^{n/2}\Gamma(n/2)}, \quad x > 0$$

Notes.
1. The Chi-Squared(n) distribution is the same as the Gamma($n/2, 1/2$) distribution.
2. The sample variance of n independent and identically distributed Normal(μ, σ^2) random variables multiplied by $\frac{n-1}{\sigma^2}$ is a Chi-Squared($n - 1$) random variable, and it is independent of the sample mean.

- **Cauchy** X is the tangent of a uniformly distributed random angle between $-\pi/2$ and $\pi/2$. Its density is

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty$$

$E[X] = 0 \quad \text{Var}(X) = \infty.$