MA 519: Introduction to Probability Theory
Qualifying Examination – January 2021

Your PUID: Scores: (1) (2) (3) (4) (Total)

This examination consists of four questions, 25 points each, totaling 100 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

1. Recall that a Gamma random variable with parameter (α, λ) has its pdf given as:

$$p(x) = \frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} \text{ for } x > 0.$$

   Let X and Y be independent Gamma random variables with parameter (α, λ) and (β, λ).

   (a) Find the distribution of $X + Y$.

   (b) Find the distribution of Y/X.

   In both cases, *derive your results. It is not sufficient just to quote some "well-known" formula or facts.* For (b), find the cdf of Y/X first and then differentiate but you will see that there is no need to actually find the cdf analytically.

2. Let X and Y be two independent, exponentially distributed random variables with parameter λ_1 and λ_2. Let $A = \min\{X, Y\}$, $B = \max\{X, Y\}$, and $C = B - A$.

   (a) Find the joint distribution between A and B and the marginals of A and B. (To be consistent, please use a and b to denote the values taken by A and B.)

   (b) Find the joint distribution between A and C and the marginal of C. Is C independent of A? (To be consistent, please use c to denote the values taken by C.)

   (c) Find the joint distribution between C and B. Is C independent of B?

   (d) What is the probability that $A = X$, i.e. $X < Y$?

3. Let $X_1, X_2, \ldots$ be a sequence of iid random variables with mean and variance μ and σ^2. Let N be a positive integer random variable with mean and variance a and b^2. Suppose N is independent of all of the X_i's. Consider the following *random* sum:

$$S_N = X_1 + X_2 + \cdots + X_N.$$

   Note that the *number* of random variables is itself random. Find the mean and variance of S_N in terms of μ, σ, a and b. *Derive your results. It is not sufficient just to quote some "well-known" formula or facts.*

   (Hint: use conditioning on the value of N. Recall also, for any random variable X, $\text{Var}(X) = EX^2 - (EX)^2$.)

4. Let $X_1, X_2, \ldots$ be a sequence of iid standard Gaussian random variables $\mathcal{N}(0,1)$.

   (a) Let $Y_1 = X_1^2$. Find the distribution of Y_1 and relate it to some common distribution. *Derive your results. It is not sufficient just to quote some "well-known" formula or facts.*

   (b) Let $Y_i = X_i^2$. Find the distribution of $Y_1 + Y_2 + \cdots + Y_n$ and related it to some common distribution.

   (c) State the Law of Large Numbers pertaining to $Y_1 + Y_2 + \cdots + Y_n$ as n tends to infinity. *Be as quantitative as possible.*

   (d) State the Central Limit Theorem pertaining to $Y_1 + Y_2 + \cdots + Y_n$ as n tends to infinity. *Be as quantitative as possible.*