MA 519 Qualifying Exam

Your PUID:______________________________

a) This examination consists of five questions, 20 points each, totaling 100 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

b) You will have 120 min. to complete the exam. Budget your time wisely!

c) If you need extra room, use the back of the pages. Just make sure I can follow your work.

1. a) If $E_1, E_2, \ldots$ are events with $P(E_n) = 1$ for every n, prove that

$$P\left(\bigcap_{n=1}^{\infty} E_n\right) = 1.$$

b) Give an example of a probability space and an uncountably infinite family of events $\{E_\alpha\}_\alpha$ such that $P(E_\alpha) = 1$ for each α but for which

$$P\left(\bigcap_{\alpha} E_\alpha\right) \neq 1.$$

(This page intentionally left blank)

(This page intentionally left blank)

2. The following problems both involve pregnancies and twins. However, each problem can be answered independently of the other.

a) About 1% of pregnancies are with twins. Approximately 11% of all pregnancies are delivered by the 37-th week, but 55% of pregnancies with twins are delivered by the 37-th week.

   Given that a pregnancy is delivered by the 37-th week what is the probability that the pregancy was for twins?

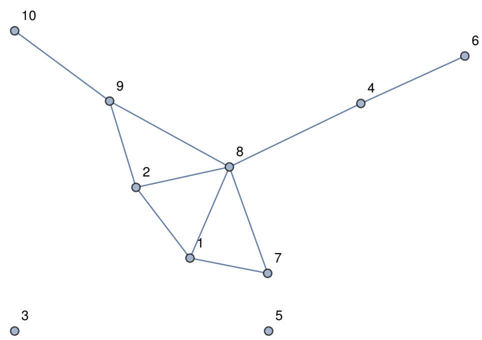
b) Suppose that 65% of pregnancies with twins have both babies of the same gender. Use this to answer the following question: given that a woman is pregnant with twins, what is the probability that the twins are identical twins? *(Explain what assumptions you are making for your probability model.)*

(This page intentionally left blank)

(This page intentionally left blank)

3. Consider the following probabilistic model for a social network. There are n individuals, and for each pair of individuals they are friends with probability $p \in (0, 1)$, independent of the frienship of any other pairs of individuals in the network.

We can represent each individual as a node on a graph labelled from 1 to n and represent "friendships" as edges between two nodes, and an example of a random network formed this way with $n = 10$ and $p = .25$ is shown below.



a) Compute the probability that there is at least one person in the network who doesn't have any friends. *Hint: use the inclusion-exclusion principle.*

b) Let $F = F_{n,p}$ be the number of friends in a random network constructed this way. Compute formulas for the mean and variance of F in terms of the parameters n and p.

(This page intentionally left blank)

(This page intentionally left blank)

4. Suppose that $X \sim \text{Poisson}(\lambda)$. That is $P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k \geq 0$.

a) Compute a formula for $E[z^X]$ for $z \in \mathbb{R}$.

b) Compute $P(X \in 2\mathbb{Z})$. *Hint: Use your formula from part (a) with* $z = -1$.

(This page intentionally left blank)

(This page intentionally left blank)

5. If X, Y are independent standard normal random variables, show that $Z = Y/X$ has a Cauchy distribution.

Recall that the density of a standard normal is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ *and the density of a Cauchy distribution is* $f(x) = \frac{1}{\pi(x^2+1)}$.

(This page intentionally left blank)

(This page intentionally left blank)