MA 519 Spring 2020
Qualifier

- You can use a calculator.
- This test is closed book and closed notes.
- You have 120 minutes.
- All problems have equal weight [10 points for each].
- Show your work.
- In order to get full credits, you need to give correct and simplified answers and explain in a comprehensible way how you arrive at them.
- **Good luck!**

Name:

Problem 1. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first. Be sure to define the sample space S corresponding to this experiment, as well as the probability $\mathbf{P}$ you are using on this sample space.

Problem 2. Let $F = \{1, 2, \ldots, n\}$ and suppose that A and B are uniformly and independently drawn among the 2^n subsets of F (including the null set $\varnothing$ and F itself) of F. **(a)** Show that $\mathbf{P}(A \subset B) = (\frac{3}{4})^n$. Hint: condition on the events $N(B) = i$, where $N(B)$ is the number of elements of B. **(b)** Show that $\mathbf{P}(AB = \varnothing) = (\frac{3}{4})^n$.

Problem 3. An urn initially contains one red and one blue ball. At each stage, a ball is randomly chosen and then replaced along with another of the same color. Let X denote the selection number of the first chosen ball that is blue. For instance, if the first selection is red and the second blue, then X is equal to 2. **(a)** Find $\mathbf{P}(X > i)$ for $i \geq 1$ **(b)** Show that, with probability 1, a blue ball is eventually chosen. **(c)** Find $\mathbf{E}[X]$.

Problem 4. An image is partitioned into two regions, one white and the other black. A reading taken from a randomly chosen point in the white section will be normally distributed with $\mu = 4$ and $\sigma^2 = 4$, whereas one taken from a randomly chosen point in the black region will have a nor- mally distributed reading with parameters $(6, 9)$. A point is randomly chosen on the image and has a reading of 5. If the fraction of the image that is black is α, for what value of α would the probability of making an error be the same, regardless of whether one concluded that the point was in the black region or in the white region?

Problem 5. (a) Let Y be a real valued continuous random variable. Prove that

$$\mathbf{E}[Y] = \int_0^\infty \mathbf{P}\left(Y > y\right)\, dy - \int_0^\infty \mathbf{P}\left(Y < -y\right)\, dy.$$

(b) For a non negative continuous random variable, variable prove that

$$\mathbf{E}\left[X^n\right] = n \int_0^\infty x^{n-1} \mathbf{P}\left(X > x\right)\, dx.$$

Problem 6. If X and Y are independent standard normal random variables, determine the joint density function of the vector (U, V), where $U = XY$ and $V = \frac{X}{Y}$. Use your result in order to show that $\frac{X}{Y}$ has a Cauchy distribution.

Values of $\Phi(x)$ for some $x \geq 0$

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936