MA 519: Introduction to Probability Theory
Qualifying Examination – August 2018

Your PUID: Scores: (1) (2) (3) (4) (5) (Total)

This examination consists of five questions, 20 points each, totaling 100 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

1. Let $S = \{1, 2, \ldots, n\}$ and suppose that A and B are independently, equally likely to be any of the 2^n subsets of S (including the empty set and the whole set S).

   (a) Find $E(N(A))$ and $Var(N(A))$, where $N(A)$ denotes the number of elements in A. (Hint: find the distribution of $N(A)$.)

   (b) Find $P(A \subseteq B)$. (Hint: condition on the value of $N(B)$)

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2. A fair dice is rolled 20 times. We say that a repeat occurs if the same number appears on consecutive rolls.

   (a) What is the expected number of repeats that occur in 20 rolls?

   (b) What is the variance of the number of repeats that occur in 20 rolls?

   (Note: For example, suppose the dice is rolled six times with an outcome pattern 455111, the (total) number of repeats is three as five repeats once and one repeats twice.

   Hint: Use the indicator function, $Y_i = 1$ if the numbers at the i-th and $(i+1)$-st rolling are the same and $Y_i = 0$ otherwise.)

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3. The pdf $p(x)$ of the Gamma distribution with parameters $(\alpha > 0, \lambda > 0)$ is given by:

$$p(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Let X and Y be independent Gamma distributed random variables with parameters (α, λ) and (β, λ). Show analytically that $X + Y$ has a Gamma distribution with parameters $(\alpha + \beta, \lambda)$.

Show how as a by-product that the above conclusion leads to the following integration identity for $\alpha, \beta > 0$:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

(You are welcome to prove the above identity by any other method.)

(Note: The function $\Gamma(\alpha)$ is defined by $\Gamma(\alpha) = \int_0^\infty x^\alpha e^{-x}\, dx$ for $\alpha > 0$.)

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4. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed continuous random variables. Let $N \geq 2$ be such that

$$X_1 > X_2 > \cdots > X_{N-1} < X_N,$$

i.e. N is the location at which the sequence stops decreasing. Prove that $E(N) = e$.

(Hint: find $P(N \geq n)$ first. Note that you do not need to worry about the distinction between $X_i > X_j$ and $X_i \geq X_j$ and so forth as the events $X_i = X_j$ for $i \neq j$ all have probability zero, due to the continuity of the random variables.)

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5. In the following, $X_1, X_2, \dots$ are i.i.d. random variables with mean $E(X_1)$ and variance $Var(X_1)$.

(a) Let n be the sample size which is a fixed deterministic number. Define the sample mean $\hat{X}$ and sample variance $\hat{\sigma}^2$ as

$$\hat{X} = \frac{1}{n}(X_1 + \cdots + X_n), \quad \hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{X})^2$$

Prove that $E(\hat{X}) = E(X_1)$ and $E(\hat{\sigma}^2) = Var(X_1)$.

(b) Now let N be a positive integer valued random number with mean $E(N)$ and variance $Var(N)$. Compute the following:

$$E\left(\sum_{i=1}^{N} X_i\right) \quad \text{and} \quad \text{Var}\left(\sum_{i=1}^{N} X_i\right).$$

Express your answers in terms of the means and variances of X_1 and N.

In both of the above parts, show your derivation clearly.

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