MA 519: Introduction to Probability Theory
January 2016, Qualifying Examination

Your PUID: ____________________________

This examination consists of six questions, 20 points each, totaling 120 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

1. Find the number of solutions, i.e. distinct r-tuples $(x_1, x_2, \ldots, x_r)$, satisfying the given conditions.

   (a) $x_1+x_2+\cdots+x_r = n$ with all the x_i's being non-negative and integral, i.e. $x_i = 0, 1, 2, \ldots$.

   (b) $x_1+x_2+\cdots+x_r \leq n$ with all the x_i's being non-negative and integral, i.e. $x_i = 0, 1, 2, \ldots$.

   (c) $1 \leq x_1 < x_2 < \cdots < x_r \leq n$ with all the x_i's being integers.

   (d) $1 \leq x_1 \leq x_2 \leq \cdots \leq x_r \leq n$ with all the x_i's being integers.

   Hint.

   (1) The following fact might be useful. Consider the equation

   $$x_1 + x_2 + \cdots x_r = n$$

   where r and n are some positive integers. The number of positive integral solutions, i.e. with all the x_i's taking values $1, 2, 3, \ldots$, is given by $\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$.

   (2) You should be able to express your answers in closed form.

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2. McDonald's newest promotion is putting a toy inside every one of its hamburger. Suppose there are N distinct types of toys and each of them is equally likely to be put inside any of the hamburger. What is the expected value and variance of the number of hamburgers you need to order (or eat) before you have a complete set of the N toys.

   (Hint: consider the number of hamburgers you need to order (or eat) in between getting one and two dinstinct types of toys, two and three distinct types of toys, and so forth. What type of random variables are they?)

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3. Let $X_1, X_2, \ldots X_n$ be n independent random variables with uniform distribution over $[0, 1]$. Let

$$Y = \min\{X_1, X_2, \ldots X_n\}, \quad Z = \max\{X_1, X_2, \ldots X_n\}, \quad M = \frac{Y+Z}{2}, \quad R = Z - Y.$$

(Well, in words, Y is the minimum, Z is the maximum, M is the mid-range and R is the range of the $x_1, x_2, \ldots, x_n$.)

(a) Find the joint pdf of Y and Z.

(b) Find the joint pdf of M and R.

(c) Find the marginal pdf of M.

(d) Find the marginal pdf of R.

(e) Are M and R independent?

Hint.

(1) for (b), it might be convenient to use change of variable formula.

(2) for (c) and (d), if you wish to use the result from (b), it might be convenient to identify pictorially the range of the point (m, r) in the mr-plane.

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4. Suppose at any time t the number of passengers arriving at a train station during the time interval $(0, t)$ is a Poisson random variable with parameter λt (i.e. λ can be interpreted as the arrival rate). Suppose also that the arrival time of the train is uniformly distributed over $(0, T)$. Assume that the arrival of passengers and the train are independent of each other.

   Find the mean and variance of the number of passengers that can board the train.

   (Hint: use conditioning on the train arrival time.)

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5. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables and N be a positive integer valued random variable independent of the X_i's. Let further

$$S = X_1 + X_2 + \cdots X_N.$$

Prove the following formula. (Assume both X and N have finite expectations and variances.)

(a) $\mathrm{E}(S) = \mathrm{E}(N)\mathrm{E}(X)$.

(b) $\mathrm{Var}(S) = \mathrm{E}(N)\mathrm{Var}(X) + (\mathrm{E}(X))^2\,\mathrm{Var}(N)$.

(Hint: use conditioning on the value of N.)

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6. Let $T_1, T_2, \ldots$ be a sequence of i.i.d. exponentially distributed random variables with parameter λ, i.e. the p.d.f. of T_1 equals $\lambda e^{-\lambda t}$ for $t \geq 0$, or in terms of its c.d.f.,

$$P(T_1 \leq t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

Let $M_n = \max_{1 \leq k \leq n} T_k$. Compute the following limits as a function of x:

(a) $\lim\limits_{n\to\infty} P(M_n \leq x)$

(b) $\lim\limits_{n\to\infty} P\left(\dfrac{M_n}{\log n} \leq x\right)$

(c) $\lim\limits_{n\to\infty} P\left(M_n - \dfrac{1}{\lambda}\log n \leq x\right)$

Explain in words (in four or five sentences) what the above results ((a), (b), and (c)) reveal about the behavior of M_n for large n, in particular the *relationship between* the results (a), (b), and (c).

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