MA 51900 PROBABILITY QUALIFYING EXAMINATION **January 2015**

Each question is worth 10 points, so the entire exam consists of 50 points.

1. Fix positive integers m and n. A person sitting at a circular table is called “isolated” if nobody is sitting to the person’s immediate left nor to the person’s immediate right. If m people sit at a table with n seats, with all seatings equally likely, find the variance of the number of people who are isolated.

2. Suppose that X and Y are jointly distributed on the triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. For $0 \le x \le 1$, the joint probability density function $f_{X,Y}(x,y)$ is proportional to x. Let W denote the product of X and Y, i.e., $W := XY$. Find the probability density function of W.

3. Suppose that a bird visits a bird feeder randomly, with interarrival times that are independent Exponential random variables, with an average of 1 visit per day. Alice checks the bird feeder, independently of the bird’s arrival times. On each day, she checks the feeder at 12:00 (noon) with probability p; or, she avoids checking the feeder that day altogether, with probability $1-p$; her decisions from day-to-day are independent. When the bird arrives, if there is food present, he immediately eats it all. Similarly, when Alice arrives, if the food is gone, she immediately refills the bird feeder. After she has just filled the bird feeder at 12:00 noon on a given day, find the expected value of the time until she fills the bird feeder again.

4. A machine consists of components A and B. All components have independent working lives, with component A’s working life having an Exponential distribution, with failure rate 1, and with component B’s working life having an Exponential distribution, with failure rate 0.10 (i.e., with expected lifetime 10). When component A fails in a machine, the failed component is replaced with a new component A. The replacement takes one unit of time, during which the machine is turned off. The component B does not age during that one unit of time (while the component A is getting replaced), since the machine is off. When component B fails, the machine scrapped. Let T be the elapsed time (including any replacement periods for A-type components) until a new machine is scrapped.
a. Find $E(T)$.
b. Find $\mathrm{Var}(T)$.

5. Suppose that $U_0, U_1, \ldots, U_n$ are independent continuous random variables, each Uniformly distributed on the interval $[0,2]$. Let T_n denote the number of sets $\{i,j\}$, with $1 \le i < j \le n$, such that $\max(U_i, U_j) < U_0$.
a. Find the probability mass function of T_n.
b. Find the expected value of T_n.
c. Find the variance of T_n.