QUALIFYING EXAMINATION

January 2009

MA 519 (M. D. Ward)

1. Each roll of a die is equally likely to show one of the six values $1, 2, \ldots, 6$.

1a. Consider a Poisson (λ) random variable N. Find the expected value and variance of the sum of the values from N rolls of a die.

1b. Roll 100 dice and let X denote the sum. Toss 700 fair coins and let Y denote the number of heads. Approximate $P(|X - Y| > 10)$.

1c. Roll 10 dice. Find the exact probability that the sum of the rolls is equal to 14.

1d. Roll a die repeatedly until a "6" appears, and stop afterwards. The jth roll of the die is considered a "record" if the value of the jth roll exceeds all of the values of the first $j-1$ rolls. Find the expected value and the variance of the number of records that occur.

1e. Roll a die repeatedly until all six values have appeared. Let R denote the total number of rolls required. Let T denote the total number of *distinct* values that appear during the first six rolls. Find the conditional expectation $E(R \mid T = 3)$.

2. Consider a collection of $n+1$ independent, continuous random variables $U_0, U_1, \ldots, U_n$, each uniformly distributed on the interval $(0, 1)$. Define $X_i = 1$ if $U_i < U_0$, and $X_i = 0$ otherwise. Define $S_n = \sum_{i=1}^{n} X_i$. Find the probability mass function of S_n.

3. Consider a group of n individuals. The individuals are (independently) each assigned a random integer value between 1 and N, with all N values equally likely. Find an exact expression for the probability that no two individuals share the same random value.

4. Fix two positive real numbers n and m with $n > m > 1$. Consider the triangle ΔABC with corners at the following three points in the plane: $A = (0, 0)$, $B = (0, n)$, $C = (m, 0)$. Consider a pair of jointly distributed continuous random variables X and Y such that (X, Y) is uniformly distributed on the triangle ΔABC. Calculate the following:

4a. The mean and variance of X.

4b. The conditional mean and conditional variance of Y, given that $X = 1$.

4c. The mean and variance of $\max(X, Y)$.

5a. Consider an exponential random variable V with mean $1/\lambda$. For each real number t, let $\lfloor t \rfloor$ denote the greatest integer less than or equal to t. Find the distribution of $\lfloor V \rfloor$. Specify the traditional name and parameter(s) of $\lfloor V \rfloor$ if you recognize them.

5b. Consider independent exponential random variables W and X, each with mean $1/\lambda$. Find the joint density of the variables $\min(W, X)$ and $|W - X|$.

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998