Score:___________

Numerical Analysis Qualifying Exam, January 2022

Note:

1. **No books and no notes in this test.**
2. **You will need a calculator for some problems in this test.**
3. **Show necessary intermediate steps of your work. No credit otherwise.**

1. (12 points)
Consider three positive floating-point numbers x_1, x_2 and x_3, where $x_1 < x_2 < x_3$. Given two ways for computing their sum:

   (i) $x_1 + x_2 + x_3$,
   (ii) $x_3 + x_2 + x_1$,

   which way can better reduce the round-off error of the result? Please explain. Suppose the computations are carried out from the left to the right.

2. (14 points)
A calculator will be needed for this problem. Determine the number of significant bits that will be lost when evaluating the expression $\left(1 - \frac{1}{x^2+1}\right)$ at $x = 0.1$.

3. (14 points)
Devise a way to compute $\frac{1}{\sqrt{7}}$ based on the Newton's method, with operations involving only additions, subtractions, or multiplications. Other mathematical operations (e.g. division, square root, etc) are not permissible. Provide the iteration formula you have devised.

4. (15 points)
Determine the coefficients a_i $(i = 0, 1, 2)$ in the third-order Adams method

   $$y_{n+1} - y_n = a_0 f_n + a_1 f_{n-1} + a_2 f_{n-2}$$

   for solving $\frac{dy}{dt} = f(t, y)$, where $f_n = f(t_n, y_n)$. Assume a step size h.

5. (15 points)
Suppose $f(x)$ $(x \in [a, b])$ has continous derivatives up to the order $(n + 1)$. Given $f(x_k)$ $(0 \leqslant k \leqslant n)$, where $x_k \in [a, b]$ are distinct from one another, show that

   $$f(x) = p(x) + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n), \quad \forall x \in [a, b]$$

   where $p(x)$ is the polynomial of the lowest degree that interpolates $f(x)$ on x_k $(0 \leqslant k \leqslant n)$, and ξ_x is some appropriate value related to $(x, x_0, x_1, \ldots, x_n)$.

6. (15 points)
   By using the interpolation polynomial and its error relation (see Problem 5), derive the trapezoidal rule, together with its error term, for approximating the intergral $\int_a^b f(x)dx$, where $f(x)$ is continuously differentiable to a sufficient order.

7. (15 points)
   Suppose $f(x)$ has continous second derivative, and it has a simple root r. Show that the secant method has the following error relation for computing r,

$$e_{n+1} = \frac{f''(\xi)}{f'(\eta)} e_n e_{n-1}$$

   where e_n is the error at step n, and ξ and η are appropriate values related to r and the approximations at steps n and $(n-1)$.