MA 514 Numerical Analysis
Qualifying Exam
August 2020

Name ________________________

PUID ________________________

INSTRUCTIONS

1. Fill in the information requested above.
2. Make sure you have a complete test. There are **7** pages (including this cover page) and **6** problems.
3. **You have to show your work in the space provided to receive partial credit.**
4. No books and no notes in this test. Calculators are allowed.
5. This is an **120** minute exam.

Question	Points	Score
1	20	
2	20	
3	15	
4	20	
5	10	
6	15	
Total:	100	

1. (20 points) To find the solution $\sqrt{a}$ to the equation $x^2 - a = 0$ $(a > 0)$, one has several ways to do the fixed point iteration:

$$x_{n+1} = \varphi(x_n), \quad n = 0, 1, 2, \ldots,$$

with

$$\varphi(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \quad \varphi(x) = \frac{a}{x}, \quad \varphi(x) = 2x - \frac{a}{x}.$$

Discuss the convergence (or nonconvergence) behavior for each of the above three iteration functions. In the case of convergence, determine the order of convergence.

2. (20 points) Suppose f is a function on the interval $[0, 3]$ for which one knows that

$$f(0) = 1, \quad f(1) = 2, \quad f'(1) = -1, \quad f(3) = f'(3) = 0.$$

(a) Find a polynomial $p(x)$ of lowest degree that satisfies the above conditions and evaluate $p(2)$.
(b) Estimate the maximum possible error of the answer given in part (a) if one knows, in addition, that $|f^{(5)}(x)| \leq M$ on $[0, 3]$. Express the answer in terms of M.

3. (15 points) Find the best least squares polynomial approximation of degree 2 to the function $f(x) = e^x$ on the interval $[0, 1]$. (Recall that the first three Legendre polynomials defined on the interval $[-1, 1]$ are $\phi_0(x) = 1$, $\phi_1(x) = x$, $\phi_2(x) = \frac{1}{2}(3x^2 - 1)$.)

4. (20 points) Determine the abscissae x_0, x_1, and weights a_0, a_1 in the Gauss quadrature

$$\int_0^\infty e^{-x} f(x)\,\mathrm{d}x \approx a_0 f(x_0) + a_1 f(x_1).$$

5. (10 points) The explicit trapezoidal method to solve the equation $y' = f(y)$ reads

$$\begin{cases} y^{(1)} = y^n + \Delta t f(y^n); \\ y^{n+1} = y^n + \dfrac{\Delta t}{2}(f(y^n) + f(y^{(1)})). \end{cases}$$

Show that this method is second order accurate.

6. (15 points) Derive the second order backward differentiation formula (BDF) for solving the equation $y' = f(y)$ (assume that the uniform time step Δt is used).