Score:___________

Numerical Analysis Qualifying Exam, August 2019

Note:

1. **No books and no notes in this test. Calculators are allowed.**
2. **Show intermediate steps of your work. no credit otherwise.**

1. (a) (6 points)
   Use the Horner's algorithm to compute the Taylor expansion of the function

   $$p(x) = x^4 - 3x^2 - 2x + 1$$

   about $x = 2$. (Note: Horner's algorithm must be used for this, no credit otherwise)

   (b) (6 points)
   Consider using the quadratic formula to find x in the following equation

   $$x^2 + 10^7 x - \frac{1}{4} = 0.$$

   Give the solution in a proper form such that no loss of significant digits will occur.

2. (a) (10 points)
   In single precision, determine the value $\left|\frac{fl(0.8)-0.8}{0.8}\right|$, where $fl(x)$ denotes the floating point number corresponding to x.

   (b) (8 points)
   Determine whether $\frac{1}{3}(1-2^{-24})$ is a machine number or not in single precision.

3. (10 points)
   We wish to compute $\frac{1}{\sqrt[4]{17}}$ on a computer. Because this computer is quite primitive, the only operations available are addition, subtraction and multiplication on it. Devise a way to compute this number based on the Newton's method, and provide the Newton iteration formula involved. Remember that only addition, subtraction and multiplication are available.

4. (12 points)
   Assume that x_i $(0 \leqslant i \leqslant n)$ are distinct points and $f(x)$ is continuously differentiable. Show that

   $$\frac{\partial}{\partial x_i} f[x_0, \ldots, x_n] = f[x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n], \quad 0 \leqslant i \leqslant n. \tag{1}$$

5. (12 points)
Prove that

$$3x = \sum_{i=-\infty}^{\infty} (t_{i+1} + t_{i+2} + t_{i+3}) B_i^3(x)$$

where $B_i^k(x)$ denotes the B-spline function of degree k having knots t_i $(i = \dots, -1, 0, 1, \dots)$.

6. (12 points)
The trapezoidal rule is given by

$$\int_a^b f(x)dx = \frac{b-a}{2}[f(a) + f(b)] - \frac{(b-a)^3}{12} f''(\xi), \quad \xi \in [a, b].$$

Derive the composite trapezoidal rule, together with its error term. Assume that $f(x)$ has continuous second derivative.

7. (12 points)
Determine the Gaussian quadrature formula of the form

$$\int_{-1}^1 x^2 f(x)dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2), \quad x_0 \leqslant x_1 \leqslant x_2.$$

8. (12 points)
$f(x)$ has continuous second derivative, and r is a simple root of this function. A root-finding method provides the following formula when solving $f(x) = 0$,

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)},$$

where n is the step index. Treating this as a fixed point iteration, use the properties of fixed point iteration to show that the convergence rate of this method is at least quadratic for computing r.